2. USING DIAGRAMS

(*a*) In Basil Brayne's class there are 33 pupils, and everyone does either French or Physics or both. Basil, who is not very bright, reckoned that he had proved that 33 = 41. He found that there were 18 doing French and 23 doing Physics. So he said:

'This makes 18 + 23, that is, 41 of us altogether. So 33 = 41.'

Explain why his argument broke down.

Penny Dropper, who is good at explaining things, said:

'You have counted some people twice. Some of us do both French and Physics. Look at it this way, Basil. If those of us who do French stood at this side of the room and those who do Physics stood over that side, then some people would not know which side to stand on. If we put them in the middle it would look like this:

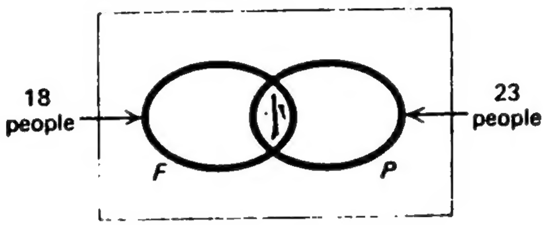


Fig. 4

Those in the middle are what we call *F* intersection *P*, or $F \cap P$.'

Basil asked her how many there would be in $F \cap P$. Penny did a quick calculation in her head and said 'Eight'.

Explain how she arrived at that number.

(*b*) To make sure that Basil understood, Penny said:

'Here is another problem. Suppose in another class there were 24 doing History, 15 doing Chemistry, and 9 doing both. How many people would there be in the class altogether, if everyone did one or other or both of these subjects?'

Basil drew a diagram like the one in Figure 5.

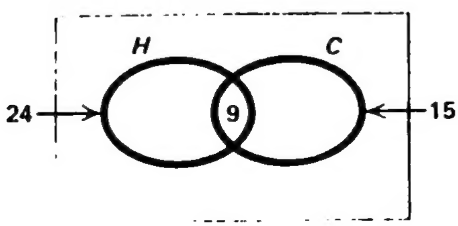


Fig. 5

'What you want to know', he said, 'is how many there are in the union of H and C'.

He thought for a moment and then said 'There are 30 in $H \cup C$'. Was he right?

(*c*) Diagrams such as those Basil and Penny have drawn are useful in problems concerned with sets of things. They illustrate some of the results you will have found from your punched cards.

The shaded part of Figure 6 shows A intersection B, ($A \cap B$),

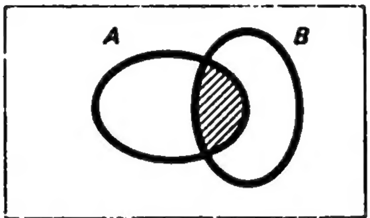


Fig. 6

and the shaded part of Figure 7 shows A union B, ($A \cup B$).

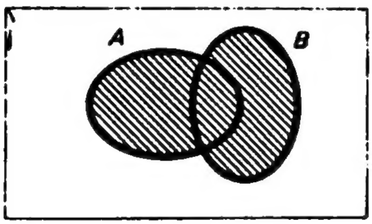


Fig. 7

Exercise A

1 In Basil's class of 33 pupils, everyone does either History or Chemistry or both. There are 24 doing History and 17 doing Chemistry. How many do both?

2 Figure 8 shows that in one class, 16 pupils do German, 18 do Woodwork and 5 do both. How many do German only? How many do Woodwork only? How many are there in the class if everyone does either German or Woodwork or both?

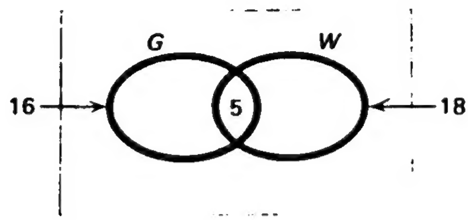


Fig. 8

3 In a class of 30 pupils there are 19 who play tennis and 16 who play hockey. If everyone plays at least one of these games how many play both?

4 Out of 17 girls in a class, 5 always walk to school, 7 use bicycles, and 9 use the bus service.

$$5+7+9 \text{ is more than } 17.$$

What is the explanation?
How many sometimes cycle and sometimes bus?

5 Draw a diagram for this information:
In a survey 100 people were questioned about the BBC TV programmes they had watched the previous day.

73 had watched BBC 1.

28 had watched BBC 2.

12 had watched both.

How many people had not watched either?

6 All the inhabitants of a certain town on the French–German border speak either French or German or both. If 64% speak French and 58% speak German, what percentage speak both languages?

7 The number of people in $A \cup B$ is 69, the number in A is 41, and the number in B is 53. Draw a diagram with two overlapping curves representing sets A and B.
How many people are there in $A \cap B$?

8 The number of people in C is 12, in D is 11, and in $C \cap D$ is 4. Draw a diagram with two overlapping curves representing sets C and D.
How many are there in $C \cup D$?

9 Invent some questions like the ones in this exercise which refer to your own class.

10 Make six rough copies of Figure 9 and on separate diagrams, shade the following sets:

(*a*) $A \cap B$; (*b*) $A \cup B$; (*c*) $B \cap C$;
(*d*) $B \cup C$; (*e*) $C \cap A$; (*f*) $C \cup A$.

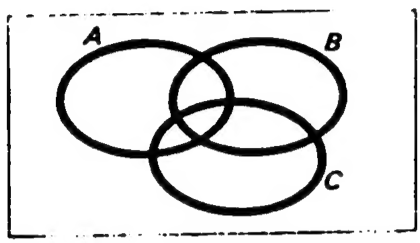


Fig. 9

3. SETS OF POINTS AND NUMBERS

We have been finding the intersection and union of sets of *people*. The same ideas apply to other sets, such as sets of points and sets of numbers. Here are some examples.

(*a*) On squared paper, mark the set of points, A, such that $x = 3$.
Mark also the set of points, B, such that $x = y$.
Indicate $A \cap B$.
This is why the word 'intersection' is used: the lines *intersect* or cross each other at the point $A \cap B$.

(*b*) On squared paper, shade the set of points, C, such that $x > 3$.
In another colour, shade the set of points, D, such that $y > 4$.
Which set on your diagram is $C \cap D$? Which set is $C \cup D$?

(*c*) Suppose E is the set of multiples of two, up to twenty, that is, E is the set 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

We usually write this as:

$$E = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}.$$

The curly brackets are short for 'is the set of'.
Give another description of F if

$$F = \{3, 6, 9, 12, 15, 18\}.$$

Write down the members of the set $E \cap F$.
Write down the members of the set $E \cup F$.

(*d*) Give another description of

(i) {a, e, i, o, u};

(ii) {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}.

For short, call (i) V and (ii) C.
What is $V \cup C$?
How many members in $V \cap C$?

Exercise B

1 Copy and complete:

(*a*) $\{1, 3, 7, 10\} \cap \{2, 3, 4, 6\} = \{\qquad\qquad\}$;

(*b*) $\{1, 3, 7, 10\} \cup \{2, 3, 4, 6\} = \{\qquad\qquad\}$;

(*c*) $\{a, b, c\} \cup \{d, e, f\} = \{\qquad\qquad\}$;

(*d*) $\{a, b, c\} \cup \{c, d, e, f\} = \{\qquad\qquad\}$;

(*e*) $\{7, 9, \quad\} \cap \{5, \quad, 2, 3\} = \{9, 3\}$.

2 On squared paper, mark the set of points, A, such that $x = 2$. Mark also the set, B, such that $y = 3$.
What can you say about $A \cap B$?

3 On squared paper, shade the set of points, C, such that $x > 4$. In another colour, shade the set of points, D, such that $y < 2$.
Which set is $C \cap D$?
Which set is $C \cup D$?

4 E is the region $x < 3$, F the region $y > 1$, and G the region $x > 5$. Using different colours shade them in.
Show on separate diagrams, the regions

(*a*) $E \cap F$;
(*b*) $F \cap G$;
(*c*) $G \cap E$;
(*d*) $E \cup F$.

5 List the members of the following sets:

H = {multiples of 2 which are less than 40},
I = {multiples of 3 which are less than 40},
J = {multiples of 5 which are less than 40}.

Find (*a*) $H \cap I$;
(*b*) $I \cap J$;
(*c*) $J \cap H$

6 List the members of the following sets:

K = {factors of 15},
L = {factors of 18},
M = {factors of 12},
N = {prime numbers less than 20}.

Find:

(*a*) $K \cap N$;
(*b*) $L \cap N$;
(*c*) $M \cap N$.

7 List the members of the following sets:

P = {prime numbers less than 50},
R = {rectangle numbers less than 50},
S = {square numbers less than 50},
T = {triangle numbers less than 50}.

Find:

(*a*) $S \cap R$;
(*b*) $S \cap T$;
(*c*) $P \cap T$;
(*d*) $R \cap T$.

8 Make two copies of Figure 10 and on one shade the set

$$A \cap (B \cup C)$$

and on the other

$$(A \cap B) \cup (A \cap C).$$

What do you notice?

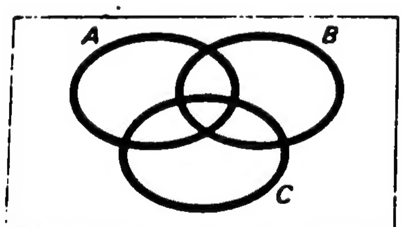


Fig. 10

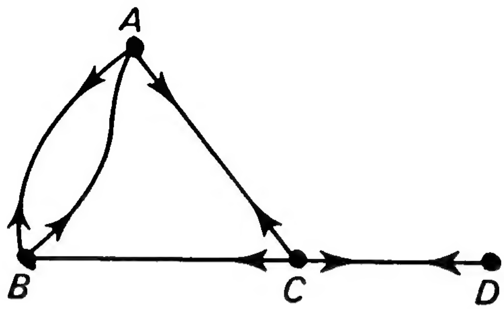
9 Make two more copies of Figure 10 and on one shade the set

$$A \cup (B \cap C)$$

and on the other

$$(A \cup B) \cap (A \cup C).$$

What do you notice?



$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3. Matrices

1. STORING INFORMATION

(*a*) Mrs Brayne orders all the milk for the four families living in her block of flats. The table below refers to the number of bottles of milk delivered one Monday.

	No. 5*A*	No. 5*B*	No. 5*C*	No. 5*D*
Gold	2	0	2	1
Red	0	2	1	3
Silver	0	1	1	1

MONDAY

On Tuesday, the families wanted:

	No. 5*A*	No. 5*B*	No. 5*C*	No. 5*D*
Gold	1	0	2	2
Red	0	3	3	2
Silver	0	1	0	2

TUESDAY

Mrs Brayne needs to know how much milk each family has. Instead of writing out charts in full like the ones above, she stores the information in number parcels. On Monday, she wrote:

$$\begin{pmatrix} 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

She knows what each number in this parcel means because, for example, the 2nd row always refers to red top milk and the 3rd column to No. 5*C*.

What number parcel did she write for Tuesday?

Number parcels are called *matrices* and we have used them before for storing information. For example, the matrix at the head of this chapter describes a network. What information does it give? What does the number 2 tell you?

(*b*) We do not know what Mrs Brayne wrote for Wednesday. But we do know that the matrix has the same shape as the others. How many rows has it? How many columns?

A matrix which has 3 *rows* and 4 *columns* is called a 3 by 4 matrix. We always write the *row number first*. We call 3 by 4 the *order* of the matrix.

(*c*) Explain why the order of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 6 & 5 & 7 \end{pmatrix} \text{ is 2 by 5.}$$

(*d*) State the orders of the following matrices:

(i) $\begin{pmatrix} 6 & 4 \\ 1 & -2 \\ 3 & 5 \\ -1 & 1 \end{pmatrix}$; (ii) $\begin{pmatrix} 1 & 3 & 0 \\ 5 & 1 & 6 \end{pmatrix}$.

(*e*) State the order of the matrix $\begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$. We sometimes call this a *column matrix* because all the numbers are in one column.

(*f*) What is the order of the matrix $(2 \quad 4 \quad -1 \quad 5)$? What name do you think we give to a matrix when all the numbers are in one row?

(*g*) What is the order of the matrix at the head of this chapter? We call it a *square matrix*. Why? Write down a square matrix which has only two columns.

Exercise A

1 Write down: (*a*) a 2 by 4 matrix;
(*b*) a 4 by 3 matrix;
(*c*) a 5 by 2 matrix;
(*d*) a 2 by 5 matrix.

2 State the orders of the following matrices:

(*a*) $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$; (*b*) $(4 \quad 6 \quad 1)$; (*c*) $\begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix}$;

(*d*) $\begin{pmatrix} 7 & 6 & 2 & 1 \\ 3 & 0 & {}^{-}5 & 9 \end{pmatrix}$; (*e*) $\begin{pmatrix} 8 & 1 & 2 \\ -1 & 3 & 5 \\ 6 & 4 & 0 \end{pmatrix}$;

(*f*) $(7 \quad 5 \quad 9 \quad 4 \quad 3)$; (*g*) $\begin{pmatrix} 2 \\ 7 \\ 6 \\ 3 \end{pmatrix}$; (*h*) $(4 \quad {}^{-}3)$.

Which of these matrices are (i) column matrices, (ii) row matrices, (iii) square matrices?

3 Three shops hold the following stocks of 'pop' records. Shop A has 60 L.P.s, 87 E.P.s and 112 singles; shop B has 103 L.P.s, 41 E.P.s and 58 singles; shop C sells only singles and has 147 of them. Write this information in matrix form and state the order of the matrix.

4 One Sunday in November the following information was obtained from a newspaper: at the top of the first division, Liverpool had played 18 games, won 11, drawn 4 and lost 3; Everton had played 18, won 10, drawn 6 and lost 2; Leeds had played 17, won 10, drawn 5 and lost 2. Write this information in a 3 by 4 matrix.

2. MATRIX ADDITION

(*a*) On Monday, Mrs Brayne ordered:

$$\begin{pmatrix} 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

and on Tuesday:

$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 3 & 3 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

How much gold top milk did each family order on Monday and Tuesday together?

Copy and complete the 3 by 4 matrix below to show how much of each type of milk each flat had on Monday and Tuesday together:

$$\begin{pmatrix} 3 & 0 & 4 & 3 \\ 0 & & & \\ 0 & & & \end{pmatrix}.$$

To complete this matrix you had to add the pairs of numbers which have corresponding positions in the matrices for Monday and Tuesday. For example, to find the bottom right-hand number, you had to add the bottom right-hand numbers 1 and 2 to obtain 3.

This way of combining two matrices to form a third is called *matrix addition* and we write:

$$\begin{pmatrix} 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 3 & 3 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 4 & 3 \\ 0 & 5 & 4 & 5 \\ 0 & 2 & 1 & 3 \end{pmatrix}.$$

We can *add* two matrices only if they have the *same order*, that is, if they are the same shape. Why?

(*b*) We often label matrices with capital letters in heavy type like this:

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 4 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \\ 4 & 2 \end{pmatrix}.$$

Then we can write

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 3+2 & 4+3 \\ 5+3 & 6+0 \\ 4+4 & 5+2 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 8 & 6 \\ 8 & 7 \end{pmatrix}.$$

What is the order of (i) $\mathbf{A}$; (ii) $\mathbf{B}$; (iii) $\mathbf{A}+\mathbf{B}$?

(*c*) We can also subtract $\mathbf{B}$ from $\mathbf{A}$ if we wish and write

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 3-2 & 4-3 \\ 5-3 & 6-0 \\ 4-4 & 5-2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 6 \\ 0 & 3 \end{pmatrix}.$$

If $$\mathbf{C} = \begin{pmatrix} 9 & 0 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix},$$

find (i) $\mathbf{C}+\mathbf{D}$; (ii) $\mathbf{D}+\mathbf{C}$; (iii) $\mathbf{C}+\mathbf{C}$; (iv) $\mathbf{C}-\mathbf{D}$.

Exercise B

1 Let $$\mathbf{P} = \begin{pmatrix} 3 & 4 & 2 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } \mathbf{Q} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}.$$

Find (*a*) $\mathbf{P}+\mathbf{Q}$; (*b*) $\mathbf{Q}+\mathbf{P}$. What do your answers suggest?

2 Write down any two matrices **A** and **B** which have the *same order*. Find (*a*) $\mathbf{A}+\mathbf{B}$; (*b*) $\mathbf{B}+\mathbf{A}$. Do you think that matrix addition is commutative?

3 Let $$\mathbf{T} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 5 & 6 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 3 & 2 \end{pmatrix} \text{ and } \mathbf{V} = \begin{pmatrix} 5 & 2 \\ 3 & 2 \\ 4 & 1 \end{pmatrix}.$$

Find (*a*) $(\mathbf{T}+\mathbf{U})+\mathbf{V}$; (*b*) $\mathbf{T}+(\mathbf{U}+\mathbf{V})$. What do your answers suggest? Is there any confusion if we write $\mathbf{T}+\mathbf{U}+\mathbf{V}$?

4 Write down any three matrices **A**, **B** and **C** which have the *same order*. Find (*a*) $(\mathbf{A}+\mathbf{B})+\mathbf{C}$; (*b*) $\mathbf{A}+(\mathbf{B}+\mathbf{C})$. Do you think that matrix addition is associative?

5 Let $$\mathbf{E} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{pmatrix}.$$

(*a*) Find $\mathbf{E}+\mathbf{E}$.

(*b*) We write $\mathbf{E}+\mathbf{E}$ as $2\mathbf{E}$. How would you write

(i) $\mathbf{E}+\mathbf{E}+\mathbf{E}$; (ii) $\mathbf{E}+\mathbf{E}+\mathbf{E}+\mathbf{E}+\mathbf{E}$?

(*c*) Find: (i) $3\mathbf{E}$; (ii) $5\mathbf{E}$.

6 Let $$\mathbf{G} = \begin{pmatrix} 11 & 3 \\ 4 & 7 \end{pmatrix} \text{ and } \mathbf{H} = \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}.$$

Find: (*a*) $\mathbf{G}+\mathbf{H}$; (*b*) $\mathbf{G}-\mathbf{H}$; (*c*) $2\mathbf{G}$;
(*d*) $4\mathbf{H}$; (*e*) $\mathbf{G}+4\mathbf{H}$; (*f*) $2\mathbf{G}-4\mathbf{H}$.

7 In the first match of the cricket season Brown bowled 12 overs, of which 4 were maidens, and he took 5 wickets for 25 runs. In the second match he took 3 wickets in 17 overs, of which 3 were maidens, for 52 runs.

Write the figures for each match as 1 by 4 matrices. (Be careful about the order in which you write the numbers!) Now write down the 1 by 4 matrix which shows Brown's figures for the two matches combined.

8 Let

$$J = \begin{pmatrix} 3 & 2 & 2 \\ 4 & 5 & 9 \end{pmatrix},\quad K = \begin{pmatrix} 4 & 5 \\ 2 & 3 \\ 6 & 7 \end{pmatrix},\quad L = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } N = \begin{pmatrix} 5 & 4 & 7 \\ 8 & 0 & 9 \end{pmatrix}.$$

Find where possible:

(*a*) $K+M$; (*b*) $J+L$; (*c*) $J+M$; (*d*) $J+N$;
(*e*) $N+J$; (*f*) $2N$; (*g*) $N+K$; (*h*) $5N$.

3. COMBINING ROW AND COLUMN MATRICES

So far we have combined matrices by adding and subtracting them. We shall now see that sometimes it is sensible to combine them in a different way.

In an athletics match between three schools, points were given for 1st, 2nd, 3rd and 4th places. This table shows the results for school *A*:

	1sts	2nds	3rds	4ths
School *A*	5	2	4	4

and this shows the number of points awarded for each place:

	1sts	2nds	3rds	4ths
Number of points	5	3	2	1

Find the total number of points gained by school *A*.

We can write both sets of information as matrices:

$$(5 \quad 2 \quad 4 \quad 4) \quad \text{and} \quad (5 \quad 3 \quad 2 \quad 1).$$

Do you agree that

$(5 \quad 2 \quad 4 \quad 4)$ can be combined with $(5 \quad 3 \quad 2 \quad 1)$ to give 43?

This way of combining matrices is called *matrix multiplication* and we usually write it like this:

$$(5 \quad 2 \quad 4 \quad 4)\begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 25+6+8+4 = 43.$$

The *first* matrix is written as a *row* matrix and the *second* matrix as a *column* matrix. It is not necessary to put brackets round 43 since it is a single element. Notice also that there is no multiplication sign between the two matrices which we wish to combine.

School B gained 0 1sts, 8 2nds, 7 3rds and 7 4ths. We can find the total number of points gained by working out

$$\begin{pmatrix} 0 & 8 & 7 & 7 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 0+24+14+7 = 45.$$

We can multiply matrices in this way only if the number of *elements* or numbers in the row matrix is equal to the number of elements in the column matrix.

We cannot write

$$\begin{pmatrix} 8 & 7 & 7 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

because it has no meaning. Why not?

School C gained 7 1sts, 2 2nds, 1 3rd, and 1 4th. Write this information as a row matrix. By multiplying two matrices together, find the total number of points gained by school C. Set your work out carefully.

Which school won the match?

Exercise C

1 Chelsea have won 10 games, drawn 6 and lost 2. We can show this by the row matrix $\begin{pmatrix} 10 & 6 & 2 \end{pmatrix}$.

For a win they get 2 points, for a draw 1 point and for losing 0 points. We can show this by the column matrix $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

Multiply the two matrices to find how many points Chelsea have.

2 Work out:

(*a*) $\begin{pmatrix} 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$;

(*b*) $\begin{pmatrix} 4 & 5 & 6 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$;

(*c*) $\begin{pmatrix} 10 & 20 \end{pmatrix}\begin{pmatrix} 20 \\ 30 \end{pmatrix}$;

(*d*) $\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$;

(*e*) $\begin{pmatrix} 5 & 2 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$;

(*f*) $\begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

3 In a traffic census the following information was obtained:

	1 in car	2 in car	3 in car	4 in car
Number of cars	96	40	20	9

Write this information as a row matrix.

Use the column matrix $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ to find the total number of people carried in all the cars.

4 The way in which the first XV scored in the first match of the season is as follows:

	Tries	Conversions .	Penalty goals
Match 1	5	1	3

3 points are awarded for a try, 2 for a conversion and 3 for a penalty goal. Write out the points scheme as a column matrix and use it and another matrix to find how many points the team scored in their first match.

	Tries	Conversions	Penalty goals
Match 2	3	2	1
Match 3	2	0	3

Do the same for the second and third matches.

5 Work out the following multiplications *if they have a meaning:*

(*a*) $(5 \quad 7)\begin{pmatrix}6\\8\end{pmatrix}$; (*b*) $(2 \quad 3 \quad 4)\begin{pmatrix}5\\6\\7\\8\end{pmatrix}$;

(*c*) $(0 \quad 0 \quad 0)\begin{pmatrix}0\\0\\0\end{pmatrix}$; (*d*) $(4 \quad 5)\begin{pmatrix}2\\3\\0\end{pmatrix}$;

(*e*) $(0 \quad 0 \quad 0 \quad 0)\begin{pmatrix}0\\0\end{pmatrix}$; (*f*) $(3 \quad 4 \quad 5 \quad 6)\begin{pmatrix}4\\3\\2\\1\end{pmatrix}$.

4. MATRIX MULTIPLICATION

We can use a 3 by 4 matrix to show the results of the athletics match for all three schools:

$$\begin{array}{r} \\ \text{School } A \\ \text{School } B \\ \text{School } C \end{array} \begin{array}{c} \begin{array}{cccc} \text{1sts} & \text{2nds} & \text{3rds} & \text{4ths} \end{array} \\ \begin{pmatrix} 5 & 2 & 4 & 4 \\ 0 & 8 & 7 & 7 \\ 7 & 2 & 1 & 1 \end{pmatrix} \end{array}.$$

We know that

$$(5 \quad 2 \quad 4 \quad 4)\begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 43,$$

so school A gained 43 points.

Also,

$$(0 \quad 8 \quad 7 \quad 7)\begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 45$$

and

$$(7 \quad 2 \quad 1 \quad 1)\begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 44,$$

so school B gained 45 points and school C gained 44 points.

We can stack these three multiplications together and write:

$$\begin{array}{r} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccc} \text{1sts} & \text{2nds} & \text{3rds} & \text{4ths} \end{array} \\ \begin{pmatrix} 5 & 2 & 4 & 4 \\ 0 & 8 & 7 & 7 \\ 7 & 2 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{r} \text{1sts} \\ \text{2nds} \\ \text{3rds} \\ \text{4ths} \end{array} \begin{array}{c} \text{points} \\ \begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix} \end{array} = \begin{array}{r} \\ A \\ B \\ C \end{array} \begin{array}{c} \text{points} \\ \begin{pmatrix} 43 \\ 45 \\ 44 \end{pmatrix} \end{array}.$$

The red labels are not necessary. They have been put in to help you to see how the three separate multiplications have been stacked together to form a single multiplication.

Copy and complete the following multiplication to find the number of points each school would have gained if 8 points had been given for a 1st, 5 points for a 2nd, 3 points for a 3rd and 1 point for a 4th.

$$\begin{pmatrix} 5 & 2 & 4 & 4 \\ 0 & 8 & 7 & 7 \\ 7 & 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 66 \\ \\ \end{pmatrix}.$$

Which school would have won the match if this scoring system had been used?

We can now stack the two scoring systems side by side and write:

$$\begin{pmatrix}5 & 2 & 4 & 4\\ 0 & 8 & 7 & 7\\ 7 & 2 & 1 & 1\end{pmatrix}\begin{pmatrix}5 & 8\\ 3 & 5\\ 2 & 3\\ 1 & 1\end{pmatrix} = \begin{pmatrix}43 & 66\\ 45 & 68\\ 44 & 70\end{pmatrix}.$$

When we multiply two matrices we combine each row of the first matrix with each column of the second matrix. Let us see this happen again for the matrices

$$\mathbf{A} = \begin{pmatrix}3 & 2 & 4\\ 4 & 2 & 1\end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix}1 & 1\\ 2 & 0\\ 1 & 3\end{pmatrix}.$$

The first row of **A** combines with the columns of **B** to give

$$(3 \quad 2 \quad 4)\begin{pmatrix}1\\ 2\\ 1\end{pmatrix} = 11 \quad \text{and} \quad (3 \quad 2 \quad 4)\begin{pmatrix}1\\ 0\\ 3\end{pmatrix} = 15.$$

The second row of **A** combines with the columns of **B** to give

$$(4 \quad 2 \quad 1)\begin{pmatrix}1\\ 2\\ 1\end{pmatrix} = 9 \quad \text{and} \quad (4 \quad 2 \quad 1)\begin{pmatrix}1\\ 0\\ 3\end{pmatrix} = 7.$$

Stacking the rows gives

$$\begin{pmatrix}3 & 2 & 4\\ 4 & 2 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\\ 1\end{pmatrix} = \begin{pmatrix}11\\ 9\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}3 & 2 & 4\\ 4 & 2 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\\ 3\end{pmatrix} = \begin{pmatrix}15\\ 7\end{pmatrix},$$

and now stacking the columns, we write

$$\mathbf{AB} = \begin{pmatrix}3 & 2 & 4\\ 4 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 0\\ 1 & 3\end{pmatrix} = \begin{pmatrix}11 & 15\\ 9 & 7\end{pmatrix}.$$

What is the order of **A**? The 3 tells us the number of elements in each row of **A**. What is the order of **B**? This time the 3 tells us the number of elements in each column of **B**. Since these two numbers are equal, we can combine the rows of **A** with the columns of **B** and work out **AB**. If the numbers were not equal, **AB** would not have a meaning.

$$\mathbf{BA} = \begin{pmatrix}1 & 1\\ 2 & 0\\ 1 & 3\end{pmatrix}\begin{pmatrix}3 & 2 & 4\\ 4 & 2 & 1\end{pmatrix}.$$

Does this have a meaning?

If $$\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 3 & 5 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 1 & 4 \\ 7 & 3 \\ 2 & 5 \end{pmatrix},$$

does CD have a meaning? Does DC have a meaning?

Copy and complete the following multiplications. The red loops in the first one are to remind you how to work out the answers. Underneath each matrix write its order.

(*a*) $$\begin{pmatrix} 2 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 8 & 14 \\ 9 & & \end{pmatrix};$$

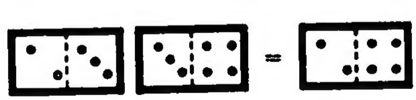
(*b*) $$\begin{pmatrix} 7 & 2 \\ 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 & 1 \\ 4 & 5 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 15 & 31 & 14 & 13 \\ & & & 1 \\ & & & 7 \end{pmatrix};$$

(*c*) $$\begin{pmatrix} 6 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 9 & \\ & 22 \end{pmatrix};$$

(*d*) $$\begin{pmatrix} 8 & 1 & 5 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} 5 & 0 & 2 & 0 \\ 1 & 4 & 1 & 0 \\ 9 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} & & 27 & 5 \\ 73 & 21 & & \end{pmatrix};$$

(*e*) $$\begin{pmatrix} 5 & 0 \\ 2 & 3 \\ 1 & 5 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 6 & 1 & 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & & & & \\ & 7 & 12 & & \\ & & 20 & 3 & \\ & & & & 13 \end{pmatrix}.$$

Look again at the orders which you have written underneath the matrices and compare them with this domino pattern:



Exercise D

1 Work out:

(a) $\begin{pmatrix}3 & 4 & 5\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}$; (b) $\begin{pmatrix}1 & 3 & 6\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}$;

(c) $\begin{pmatrix}3 & 4 & 5\\1 & 3 & 6\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}$; (d) $\begin{pmatrix}3 & 4 & 5\\1 & 3 & 6\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 5\\3 & 4\end{pmatrix}$.

2 Work out:

(a) $\begin{pmatrix}7 & 1 & 5\end{pmatrix}\begin{pmatrix}2\\8\\0\end{pmatrix}$; (b) $\begin{pmatrix}7 & 1 & 5\end{pmatrix}\begin{pmatrix}3\\6\\2\end{pmatrix}$;

(c) $\begin{pmatrix}7 & 1 & 5\end{pmatrix}\begin{pmatrix}2 & 3\\8 & 6\\0 & 2\end{pmatrix}$; (d) $\begin{pmatrix}7 & 1 & 5\\0 & 4 & 3\\1 & 5 & 6\end{pmatrix}\begin{pmatrix}2 & 3\\8 & 6\\0 & 2\end{pmatrix}$.

3 Work out:

(a) $\begin{pmatrix}2 & 1 & 0\\3 & 2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\3 & 0\\4 & 1\end{pmatrix}$; (b) $\begin{pmatrix}3 & 4\\5 & 6\end{pmatrix}\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}$;

(c) $\begin{pmatrix}4 & 5 & 6 & 0\\0 & 1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 3\\2 & 2\\3 & 1\\4 & 0\end{pmatrix}$; (d) $\begin{pmatrix}2 & 3\\1 & 5\\6 & 0\end{pmatrix}\begin{pmatrix}5 & 0 & 6\\2 & 7 & 3\end{pmatrix}$;

(e) $\begin{pmatrix}3 & 0 & 1\\0 & 5 & 4\\2 & 1 & 5\end{pmatrix}\begin{pmatrix}6 & 0 & 1\\2 & 3 & 4\\3 & 2 & 1\end{pmatrix}$; (f) $\begin{pmatrix}1 & 2\\0 & 5\\3 & 1\\4 & 2\end{pmatrix}\begin{pmatrix}6 & 3 & 7\\0 & 4 & 1\end{pmatrix}$.

4 Let

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 5 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

(*a*) Write down the orders of the matrices **A**, **B**, **C** and **D**.
(*b*) We can find **BC**. Why? Find **BC**.
(*c*) Can we find **DA**? Give a reason for your answer. Find **DA** if possible.
(*d*) Can we find **AB**? Give a reason for your answer. Find **AB** if possible.

5 (*a*) When a 3 by 4 matrix is multiplied by a 4 by 2 matrix, what is the order of the answer? (Remember the domino pattern.)
(*b*) When a 2 by 5 matrix is multiplied by a 5 by 7 matrix, what is the order of the answer?

6 Let

$$\mathbf{E} = \begin{pmatrix} 3 & 5 \\ 2 & 6 \\ 1 & 9 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 5 & 6 & 3 & 8 \\ 1 & 2 & 1 & 2 \\ 4 & 0 & 7 & 5 \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} 2 & 7 & 0 & 9 \\ 1 & 3 & 8 & 6 \end{pmatrix} \text{ and } \mathbf{H} = \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix}.$$

(*a*) What is the order of matrix **X** if **EX** = **F**?
(*b*) What is the order of matrix **Y** if **YF** = **G**?
(*c*) What is the order of matrix **Z** if **EZ** = **H**?

7 Each week the orders of three houses from a baker are as follows:

	White loaves	Wholemeal loaves	Hovis
No. 1	1	1	0
No. 2	0	1	2
No. 3	1	2	1

White bread costs 7p a loaf, wholemeal 6p and Hovis 5p. Write the prices as a *column matrix*.

Multiply two matrices together to find the total bread bill for each house.

8 'Bildit' is a constructional toy with standard parts called flats, pillars, blocks, rods and caps. It is boxed in sets, numbered 1 to 3. Set 1 has 1 flat, 4 pillars, 8 blocks, 14 rods and 2 caps. Set 2 has 2 flats, 10 pillars, 12 blocks, 30 rods and 4 caps. Set 3 has 4 flats, 24 pillars, 30 blocks, 60 rods and 10 caps. Tabulate this information in a 3 by 5 matrix.

Flats cost 5p each, pillars 4p, blocks 1p, rods 2p, and caps 3p. Write a suitable matrix for these prices and multiply two matrices together to find the cost of the various sets.

9 A factory produces three types of portable radio set called Audio 1, Audio 2 and Audio 3. Audio 1 contains 1 transistor, 10 resistors and 5 capacitors, Audio 2 contains 2 transistors, 18 resistors and 7 capacitors and Audio 3 contains 3 transistors, 24 resistors and 10 capacitors. Arrange this information in a matrix with sets in columns and parts in rows.

Find the factory's weekly consumption of transistors, resistors and capacitors if its weekly output of sets is 100 of Audio 1, 250 of Audio 2 and 80 of Audio 3.

10 $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.

Work out: (*a*) **AB**; (*b*) **BA**; (*c*) **BC**; (*d*) **CB**. What do your answers tell you about matrix multiplication?

11 $\mathbf{P} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 5 & 4 \\ 1 & 0 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}$.

Work out: (*a*) (**PQ**)**R**; (*b*) **P**(**QR**). Make sure you write the matrices in the right order! Is there any confusion if we write **PQR**?

12 Write down any three matrices **A**, **B** and **C** each of order 2 by 2. Work out: (*a*) (**AB**)**C**; (*b*) **A**(**BC**). Do you think that matrix multiplication is associative?